

# Harold's Differential Equation Models

## Cheat Sheet

23 March 2026

### 1. Exponential Growth and Decay

#### Observations:

- Population (Exponential): A population of bacteria grows at a rate directly proportional to the current population size.
- Finance: The more money you have invested the faster it grows.

#### Application:

- Bacteria growth in a Petri dish (+ $k$ )
- Compounded interest in a savings account
- Half-life of radioactive Uranium ( $-k$ )

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

Separate variables and integrate:

$$\frac{dP}{P} = k dt$$

$$\int \frac{dP}{P} = \int k dt$$

$$\ln|P| = kt + c$$

Solve for  $P(t)$ :

$$e^{\ln|P|} = e^{kt+c}$$

$$|P| = e^c e^{kt} = C e^{kt}$$

At  $t = 0$  (initial condition):

$$P(0) = C e^{k \cdot 0} = C \cdot 1 = C = P_0$$

Therefore:

For Population:

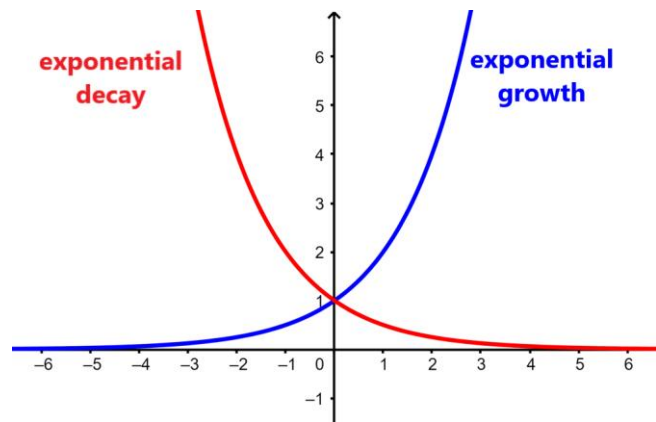
$$P(t) = P_0 e^{kt}$$

For Finance:

$$A = P e^{rt}$$

Let:

- $dP/dt$ : The instantaneous rate of change of  $P$  over time.
- $\propto$ : Symbol for "is proportional to".
- $P$ : The quantity whose rate of change is being considered. Population, principle.



Grows exponentially if  $k$  is positive ( $k > 0$ ).

Decays exponentially if  $k$  is negative ( $k < 0$ ).

## 2. Newton's Law of Cooling

### Observation:

- The rate at which an object cools or heats is directly proportional to the temperature difference between the object and its surroundings.

### Discipline:

- Thermal physics

### Application:

- Cooling/heating of objects
- Coffee temperature, forensic time-of-death estimates
- Electronics thermal modeling

$$\frac{dT}{dt} \propto \Delta T$$

$$\frac{dT}{dt} \propto T_a - T$$

$$\frac{dT}{dt} = k(T_a - T)$$

$$\frac{dT}{dt} = -k(T - T_a)$$

Separate variables:

$$\frac{dT}{T - T_a} = -k dt$$

Integrate both sides:

$$\int \frac{1}{T - T_a} dT = \int -k dt$$

$$\ln(T - T_a) = -kt - c$$

Solve for  $T(t)$ :

$$e^{\ln(T - T_a)} = e^{-kt - c}$$

$$T - T_a = e^{-c} e^{-kt} = C e^{-kt}$$

$$T(t) = T_a + C e^{-kt}$$

At  $t = 0$  (initial condition):

$$T(0) = T_a + C = T_0$$

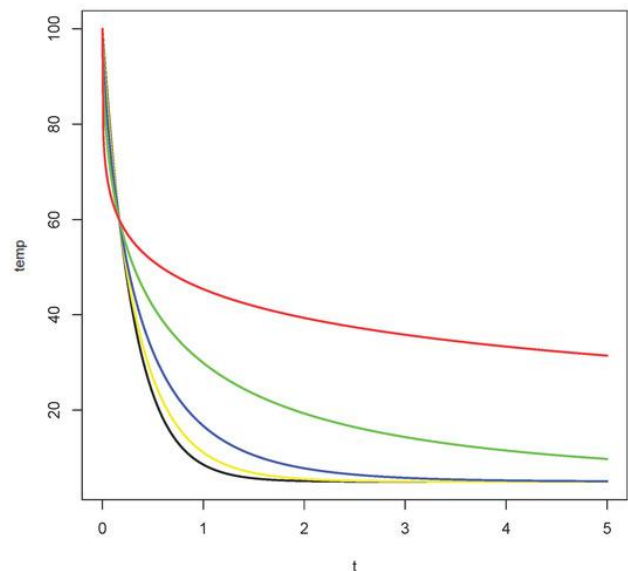
$$C = T_0 - T_a$$

Therefore:

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

Let:

- $T(t)$ : Represents the temperature of the object at time  $t$ .
- $T_a$ : Represents the constant ambient temperature of the surroundings.
- $T_0$ : Represents the initial temperature of the object.
- $k$ : Represents a positive constant, often referred to as the cooling constant, which depends on the properties of the object and the surrounding medium.



The solution shows that the object's temperature  $T(t)$  approaches the ambient temperature  $T_a$  exponentially over time.

### 3. Doomsday Differential Equation

#### Observation:

- **Population (Logistic):** The "doomsday differential equation," or "doomsday-extinction model," is a differential equation that models population growth. It can lead to either population extinction or a finite "doomsday" time where the population explodes to infinity.

$$\frac{dP}{dt} \propto P(M - P)$$

$$\frac{dP}{dt} = kP(M - P)$$

Separate variables:

$$\frac{dP}{P(M - P)} = k dt$$

Partial fraction decomposition:

$$\frac{1}{P(M - P)} = \frac{1}{M} \frac{1}{P} + \frac{1}{M - P}$$

Substitute back in and multiply both sides by  $M$ :

$$\frac{1}{P} dP + \frac{1}{M - P} dP = kM dt$$

Integrate both sides:

$$\int \frac{1}{P} dP + \int \frac{1}{M - P} dP = \int kM dt$$

$$\ln |P| - \ln |M - P| = kMt + c$$

$$\ln \left( \frac{P}{M - P} \right) = kMt + c$$

Solve for  $P(t)$ :

$$e^{\ln \left( \frac{P}{M - P} \right)} = e^{kMt + c}$$

$$\frac{P}{M - P} = e^c e^{kMt} = C e^{kMt}$$

Several algebra steps later:

$$P(t) = \frac{MC}{1 + C e^{-kMt}}$$

At  $t = 0$  (initial condition):

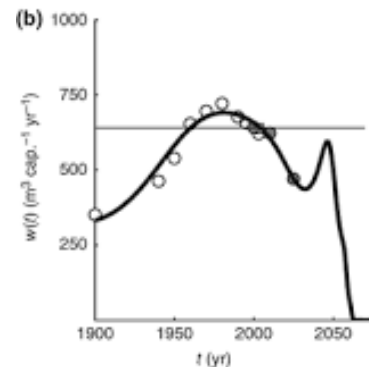
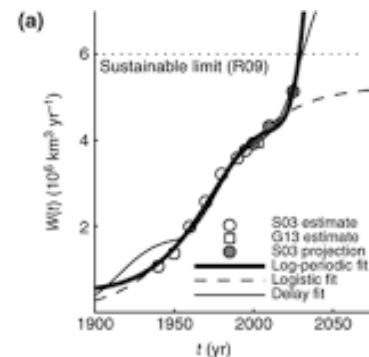
$$P(0) = P_0 \rightarrow C = \frac{P_0}{M - P_0}$$

Replacing  $C$  and more algebra steps give:

$$P(t) = \frac{M}{1 + \left( \frac{M - P_0}{P_0} \right) e^{-kMt}}$$

Let:

- $P(t)$  represents the population size at time  $t$ .
- $k$  is a positive constant related to the growth rate ( $k > 0$ ).
- $M$  is a constant representing a carrying capacity or a threshold.



#### Doomsday vs. Extinction:

- **Extinction:** If the initial population  $P(0) = P_0$  is less than  $M$ , the population will decline over time and eventually approach zero (extinction).
- **Doomsday:** If the initial population  $P(0) = P_0$  is greater than  $M$ , the population will initially grow, but it will eventually reach a point where it explodes to infinity in a finite amount of time (doomsday).

If  $P_0 > M$

$$t_{\text{doomsday}} = \left( \frac{1}{kM} \right) \ln \left( \frac{P_0}{P_0 - M} \right)$$

## 4. Drug Concentration in Body

### Observation:

- The absorption of medicine into the body can be modeled using differential equations that describe how the concentration of the drug changes over time. A first-order absorption model assumes the rate of absorption is proportional to the amount of drug available at the absorption site (e.g., in the gastrointestinal tract (GI)).

### 1. Absorption Phase by the GI Track:

The drug leaves the absorption site at a rate proportional to how much is left there.

$$\frac{dA(t)}{dt} \propto A(t)$$

$$\frac{dA(t)}{dt} = -k_a A(t)$$

$$A(t) = D e^{-k_a t}$$

### 2. Entry into Bloodstream:

The drug concentration in the bloodstream increases due to absorption and decreases due to elimination.

$$\frac{dC(t)}{dt} = \text{Absorbed} - \text{Eliminated}$$

$$\frac{dC(t)}{dt} = k_a A(t) - k_e C(t)$$

Substitute  $A(t)$  into the  $C(t)$  differential equation:

$$\frac{dC(t)}{dt} = k_a D e^{-k_a t} - k_e C(t)$$

Solving for  $C(t)$  (see next page) gives the solution for the amount of drug in the bloodstream over time:

### Multiple Oral Doses Model:

$$C(t) = \frac{k_a D}{k_e - k_a} (e^{-k_a t} - e^{-k_e t}) + C_0 e^{-k_e t}$$

### Single Oral Dose Model:

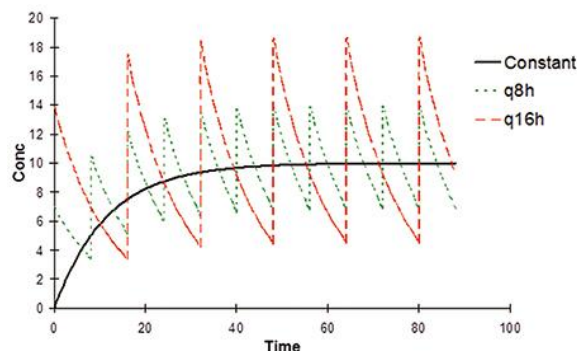
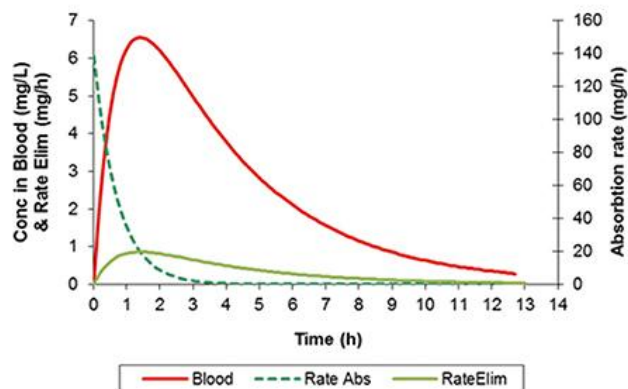
$$A(0) = D$$

$$C(0) = C_0 = 0$$

$$C(t) = \frac{k_a D}{k_e - k_a} (e^{-k_a t} - e^{-k_e t})$$

Let:

- $A(t)$ : Amount of drug at the absorption site at time  $t$
- $C(t)$ : Amount of drug (concentration) in the bloodstream at time  $t$
- $C_0$ : Existing concentration before next dose
- $t$ : Time in hours
- $D$ : Dose of the drug in milligrams
- $k_a$ : Constant absorption rate of the drug
- $k_e$ : Constant elimination rate of the drug



Interpretation:

- The first term is from absorption of the new dose.
- The second term is the decay of the existing concentration  $C_0$  due to elimination.
- If  $C_0 = 0$ , we get the standard single-dose equation.

**Solution:**

Differential equation:

$$\frac{dC(t)}{dt} = k_a D e^{-k_a t} - k_e C(t)$$

Move all C(t) variables to the left side:

$$\frac{dC(t)}{dt} + k_e C(t) = k_a D e^{-k_a t}$$

Multiply both sides by the integrating factor,  $e^{k_e t}$ :

$$e^{k_e t} \frac{dC(t)}{dt} + k_e e^{k_e t} C(t) = k_a D e^{(k_e - k_a)t}$$

The left-hand side is the product rule derivative of:

$$\frac{d}{dt} (e^{k_e t} C(t)) = k_a D e^{(k_e - k_a)t}$$

Integrate:

$$\int \frac{d}{dt} [e^{k_e t} C(t)] dt = \int k_a D e^{(k_e - k_a)t} dt$$

$$e^{k_e t} C(t) = k_a D \int e^{(k_e - k_a)t} dt + K$$

Assume  $k_a \neq k_e$  to avoid division by zero (0):

$$e^{k_e t} C(t) = \frac{k_a D}{k_e - k_a} e^{(k_e - k_a)t} + K$$

Solve for C(t):

$$C(t) = \frac{k_a D}{k_e - k_a} e^{-k_a t} + K e^{-k_e t}$$

Apply initial condition  $C(0) = C_0$ :

$$C_0 = \frac{k_a D}{k_e - k_a} + K$$

$$K = C_0 - \frac{k_a D}{k_e - k_a}$$

Therefore:

$$C(t) = \frac{k_a D}{k_e - k_a} (e^{-k_a t} - e^{-k_e t}) + C_0 e^{-k_e t}$$

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### Logistic Growth (Verhulst Equation)

**Application:**

Population biology, epidemiology, resource-limited systems

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### The SIR Epidemic Model

**Application:**

Epidemiology

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### The Lorenz System

**Application:**

Chaos theory, atmospheric science

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### Lotka–Volterra Predator–Prey Model

**Application:**

Controlling and determining the dynamics of wolves and rabbits population.

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### Accounting for wind resistance when throwing a baseball

**Application:**

A baseball follows a parabolic trajectory when thrown.  
How do we account for wind resistance to determine exactly where a ball will land?

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