

Harold's Infinite Series

Cheat Sheet

27 October 2025

| Summation Form | Expanded Form |
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| Exponential Functions | |
| $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \dots$ |
| $a^x = e^{x \ln(a)} = \sum_{n=0}^{\infty} \frac{[x \ln(a)]^n}{n!} \text{ for all } x$ | $1 + x \ln(a) + \frac{[x \ln(a)]^2}{2!} + \frac{[x \ln(a)]^3}{3!} + \dots$ |

| Natural Logarithm Functions | |
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| $\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } x < 1$ | $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \dots$ |
| $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-1)^n}{n} \text{ for } x < 1$ | $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$ |
| $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \text{ for } x < 1$ | $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \dots$ |
| $\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{2}{2n-1} x^{2n-1} \text{ for } x < 1$ | $2x - \frac{2x^3}{3} + \frac{2x^5}{5} - \frac{2x^7}{7} + \frac{2x^9}{9} - \frac{2x^{11}}{11} + \frac{2x^{13}}{13} - \dots$ |

| Geometric Series | |
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| $\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \text{ for } 0 < x < 2$ | $1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots$ |
| $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \text{ for } x < 1$ | $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - \dots$ |
| $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } x < 1$ | $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \dots$ |
| $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ for } x < 1$ | $1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + \dots$ |
| $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \text{ for } x < 1$ | $1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12} + x^{14} + \dots$ |
| $\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1} \text{ for } x < 1$ | $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - \dots$ |

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| $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \text{ for } x < 1$ | $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \dots$ |
| $\frac{1}{(1+x)^3} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (n-1)n}{2} x^{n-2} \text{ for } x < 1$ | $1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + 28x^6 - \dots$ |
| $\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2} \text{ for } x < 1$ | $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + \dots$ |
| $\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (1-2n)} x^n \text{ for } -1 < x \leq 1$ | $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1,024}x^5 + \dots$ |
| $\sqrt{1-x} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (1-2n)} x^n \text{ for } -1 < x \leq 1$ | $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \frac{21}{1,024}x^5 - \dots$ |
| $\sqrt{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (1-2n)} x^{2n} \text{ for } -1 < x \leq 1$ | $1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \frac{7}{256}x^{10} - \dots$ |
| $\sqrt{1-x^2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (1-2n)} x^{2n} \text{ for } -1 < x \leq 1$ | $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \dots$ |
| Double Factorial (!!) | $(n)!! = n(n-2)(n-4) \dots 6 \cdot 4 \cdot 2 \text{ if even}$ $(n)!! = n(n-2)(n-4) \dots 5 \cdot 3 \cdot 1 \text{ if odd}$ where $0!! = 1$ and $-1!! = 1$ |
| $\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} x^n \text{ for } -1 < x \leq 1$ | $1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \dots$ |
| $\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n \text{ for } -1 < x \leq 1$ | $1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots$ |
| $\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} x^{2n} \text{ for } -1 < x \leq 1$ | $1 - \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^8 - \dots$ |
| $\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n} \text{ for } -1 < x \leq 1$ | $1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^8 + \dots$ |

| Binomial Series | |
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| $(1+x)^r = \sum_{n=0}^{+\infty} \binom{r}{n} x^n$ <p>for $x < 1$ and all complex r where</p> $\binom{r}{n} = \frac{n!}{r!(n-r)!} = \prod_{k=1}^n \frac{r-k+1}{k}$ $= \frac{r(r-1)(r-2)\dots(r-n+1)}{n!}$ | $1 + rx + \frac{r(r-1)}{2!} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \dots$ |

| Trigonometric Functions | |
|---|---|
| $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ for all } x$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \dots$ |
| $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \text{ for all } x$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \dots$ |
| $\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}$ <p>for $x < \frac{\pi}{2}$</p> | $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1,382}{155,925}x^{11} + \frac{21,844}{608,1075}x^{13} + \dots$ |
| $\sec(x) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}$ <p>for $x < \frac{\pi}{2}$</p> | $1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + 1,385\frac{x^8}{8!} + 50,521\frac{x^{10}}{10!} + \dots$ |
| $\csc(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 2(2^{2n-1} - 1) B_{2n}}{(2n)!} x^{2n-1}$ <p>for $0 < x < \pi$</p> | $\frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15,120}x^5 + \frac{127}{604,800}x^7 + \dots$ |
| $\cot(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1}$ <p>for $0 < x < \pi$</p> | $\frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{189}x^5 - \frac{1}{4,725}x^7 - \frac{4}{2,835}x^9 - \dots$ |

| Inverse Trigonometric Functions | |
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| $\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1}$ <p>for $x \leq 1$</p> $\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} (2n+1) n!} x^{2n+1}$ | $x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7x^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} + \dots$ |
| $\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$ <p>for $x \leq 1$</p> | $\frac{\pi}{2} - x - \frac{x^3}{2 \cdot 3} - \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} - \dots$ |

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| $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$ <p style="text-align: center;">for $x < 1$</p> | $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots \text{ for } -1 < x < 1$ $\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \dots \text{ for } x \geq 1$ $-\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \dots \text{ for } x < -1$ |
| $\sec^{-1}(x) = -i \ln(x) + i \ln(2) - \frac{i}{4} \sum_{n=0}^{\infty} \frac{(2n+1)! x^{2n+2}}{4^n [(n+1)!]^2}$ | $-i \ln(x) + i \ln(2) - \frac{i}{4} x^2 - \frac{3i}{32} x^4 - \frac{5i}{96} x^6 - \dots$ |
| $\csc^{-1}(x) = i \ln(x) - i \ln(2) + \frac{\pi}{2} + \frac{i}{4} \sum_{n=0}^{\infty} \frac{(2n+1)! x^{2n+2}}{4^n [(n+1)!]^2}$ | $i \ln(x) - i \ln(2) + \frac{\pi}{2} + \frac{i}{4} x^2 + \frac{3i}{32} x^4 + \frac{5i}{96} x^6 + \dots$ |
| $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$ <p style="text-align: center;">for $x < 1$</p> | $\frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \dots \text{ for } -1 < x < 1$ $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \frac{1}{9x^9} - \dots \text{ for } x \geq 1$ $\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \frac{1}{9x^9} - \dots \text{ for } x < -1$ |

| Hyperbolic Functions | |
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| $\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x$ | $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + \dots$ |
| $\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \text{ for all } x$ | $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \frac{x^{14}}{14!} + \dots$ |
| $\tanh(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1}$ <p style="text-align: center;">for $x < \frac{\pi}{2}$</p> | $x - 2 \frac{x^3}{3!} + 16 \frac{x^5}{5!} - 272 \frac{x^7}{7!} + 7,936 \frac{x^9}{9!} - 353,792 \frac{x^{11}}{11!} + \dots$ $x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \frac{62}{2,835} x^9 - \frac{1,382}{155,925} x^{11} + \dots$ |
| $\operatorname{sech}(x) = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}$ <p style="text-align: center;">for $x < \frac{\pi}{2}$</p> | $1 - \frac{x^2}{2!} + 5 \frac{x^4}{4!} - 61 \frac{x^6}{6!} + 1,385 \frac{x^8}{8!} - 50,521 \frac{x^{10}}{10!} + \dots$ |
| $\operatorname{csch}(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(2-2^{2n}) B_{2n}}{(2n)!} x^{2n-1}$ <p style="text-align: center;">for $0 < x < \pi$</p> | $\frac{1}{x} - \frac{1}{6} x + \frac{7}{360} x^3 - \frac{31}{15,120} x^5 + \frac{127}{604,800} x^7 - \dots$ |
| $\operatorname{coth}(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} B_{2n}}{(2n)!} x^{2n-1}$ <p style="text-align: center;">for $0 < x < \pi$</p> | $\frac{1}{x} + \frac{1}{3} x - \frac{1}{45} x^3 + \frac{2}{945} x^5 - \frac{1}{4,725} x^7 + \frac{4}{2,835} x^9 - \dots$ |

| Inverse Hyperbolic Functions | |
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| $\sinh^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1}$ <p>for $x \leq 1$</p> $\sinh^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n+1)(2n)!!} x^{2n+1}$ | $x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7x^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} - \dots$ |
| $\cosh^{-1}(x) = \frac{\pi}{2}i - i \sum_{n=0}^{\infty} \frac{2^{-n}}{n! (2n+1)} x^{2n+1}$ <p>for $x \leq 1$</p> | $\frac{\pi i}{2} - ix - \frac{ix^3}{2 \cdot 3} - \frac{i \cdot 1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{i \cdot 1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} - \dots$ |
| $\tanh^{-1}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ <p>for $x < 1, x \neq \pm 1$</p> | $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \dots$ |
| $\operatorname{sech}^{-1}(x) = -\ln(x) + \ln(2)$ $+ \frac{1}{4} \sum_{n=0}^{\infty} \frac{(2n+1)! x^{2n+2}}{4^n [(n+1)!]^2}$ | $-\ln(x) + \ln(2) + \frac{1}{4}x^2 + \frac{3i}{32}x^4 + \frac{5i}{96}x^6 + \dots$ |
| $\operatorname{csch}^{-1}(x) = -\ln(x) + \ln(2)$ $+ \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)! x^{2n+2}}{4^n [(n+1)!]^2}$ | $-\ln(x) + \ln(2) + \frac{1}{4}x^2 - \frac{3i}{32}x^4 + \frac{5i}{96}x^6 - \dots$ |
| $\operatorname{coth}^{-1}(x) = -\frac{i\pi}{2} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ | $= -\frac{i\pi}{2} + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \dots$ |

| Constant | Series Summation | Expanded |
|--------------------|---|---|
| 1 | Telescoping $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ | $\left(\frac{1}{1 \cdot 2}\right) + \left(\frac{1}{2 \cdot 3}\right) + \left(\frac{1}{3 \cdot 4}\right) + \left(\frac{1}{4 \cdot 5}\right) + \left(\frac{1}{5 \cdot 6}\right) + \left(\frac{1}{6 \cdot 7}\right) + \dots$ |
| 2 | Geometric $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ | $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \dots$ |
| $\frac{1}{3}$ | Archimède $\sum_{n=1}^{\infty} \frac{1}{4^n}$ | $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1,024} + \frac{1}{4,096} + \frac{1}{16,384} + \dots$ |
| π | Nilakantha (15th Century) II $3 + \sum_{n=2}^{\infty} \frac{(-1)^n 4}{((2n-1)^3 - (2n-1))}$ | $3 + \left(\frac{4}{3^3 - 3}\right) - \left(\frac{4}{5^3 - 5}\right) + \left(\frac{4}{7^3 - 7}\right) - \left(\frac{4}{9^3 - 9}\right) + \dots$ |
| | $3 + \sum_{n=2}^{\infty} \frac{(-1)^n (4)}{(n)(n+1)(n+2)}$ | $3 + \left(\frac{4}{2 \cdot 3 \cdot 4}\right) - \left(\frac{4}{3 \cdot 4 \cdot 5}\right) + \left(\frac{4}{4 \cdot 5 \cdot 6}\right) - \left(\frac{4}{5 \cdot 6 \cdot 7}\right) + \dots$ |
| $\frac{1}{\pi}$ | Ramanujan-Sato $\frac{2\sqrt{2}}{99^2} \sum_{n=0}^{\infty} \frac{(4n)! 26,390n + 1103}{n!^4 396^{4n}}$ | $\frac{2\sqrt{2}}{99^2} \left(1103 + \frac{659,832}{396^4} + \frac{135,785,160}{396^8} + \dots\right)$ |
| $\frac{\pi}{4}$ | James Gregory (or Leibniz) $\sum_{n=1}^{\infty} \frac{(-1)^{i+1}}{(2n-1)}$ | $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \dots$ |
| | $\sum_{n=0}^{\infty} \frac{\sin[(2n+1)\theta]}{(2n+1)}$ where $0 < \theta < \pi$ | $\frac{\sin(\theta)}{1} + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + \frac{\sin(7\theta)}{7} + \dots$ |
| $\frac{\pi}{16}$ | Nilakantha (15th Century) I $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^5 + 4 \cdot (2n-1)}$ | $\left(\frac{1}{1^5 + 4 \cdot 1}\right) + \left(\frac{1}{3^5 + 4 \cdot 3}\right) + \left(\frac{1}{5^5 + 4 \cdot 5}\right) + \dots$ |
| $\frac{\pi^2}{6}$ | Basel Problem with Zeta $\zeta(2)$ (solved by Leonhard Euler) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ | $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \dots$ |
| $\frac{\pi^2}{8}$ | Leonhard Euler $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ | $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} + \dots$ |
| $\frac{\pi^2}{12}$ | Euler's Alternating $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ | $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \dots$ |

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|--------------------------|---|--|
| $\frac{\pi^2}{32}$ | Euler's Alternating $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$ | $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{17^3} - \dots$ |
| $\frac{\pi^4}{90}$ | Zeta $\zeta(4)$ $\sum_{n=1}^{\infty} \frac{1}{n^4}$ | $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \frac{1}{9^4} + \dots$ |
| $\frac{\pi^6}{945}$ | Zeta $\zeta(6)$ $\sum_{n=1}^{\infty} \frac{1}{n^6}$ | $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \frac{1}{8^6} + \frac{1}{9^6} + \dots$ |
| $\sqrt{\frac{\pi e}{2}}$ | $\sqrt{\frac{\pi e}{2}}$ | $1 + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \dots$ $+ \frac{1}{1 + \frac{1}{2}}$ $1 + \frac{1}{1 + \frac{2}{3}}$ $1 + \frac{1}{1 + \dots}$ |
| e | $\sum_{n=0}^{\infty} \frac{1}{n!}$ | $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \dots$ |
| $\frac{1}{e}$ | $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ | $\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \dots$ |
| $\ln(2)$ | Alternating Harmonic $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ | $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots$ |
| | $\sum_{n=1}^{\infty} \frac{1}{2^n n}$ | $\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} + \dots$ |
| | $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$ | $\left(\frac{1}{1 \cdot 2}\right) + \left(\frac{1}{3 \cdot 4}\right) + \left(\frac{1}{5 \cdot 6}\right) + \left(\frac{1}{7 \cdot 8}\right) + \left(\frac{1}{9 \cdot 10}\right) + \dots$ |

| Bernoulli Numbers | Euler Numbers | Gamma Function |
|--|--|--|
| $B_0 = 1$ $B_1 = -\frac{1}{2}$ $B_2 = \frac{1}{6}$ $B_4 = -\frac{1}{30}$ $B_6 = \frac{1}{42}$ $B_8 = -\frac{1}{30}$ $B_{10} = \frac{5}{66}$ $B_{12} = -\frac{691}{2,730}$ $B_{14} = \frac{7}{6}$ $B_{16} = -\frac{3,617}{510}$ $B_{18} = \frac{438,675}{798}$ $B_{20} = -\frac{174,611}{330}$ $B_{22} = \frac{854,513}{138}$ | $E_0 = 1$ $E_1 = 0$ $E_2 = -1$ $E_3 = 0$ $E_4 = 5$ $E_5 = 0$ $E_6 = -61$ $E_8 = 1,385$ $E_{10} = -50,521$ $E_{12} = 2,702,765$ $E_{14} = -199,360,981$ $E_{16} = 19,391,512,145$ $E_{18} = -2,404,879,675,441$ $E_{20} = 370,371,188,237,525$ $E_{22} = -69,348,874,393,137,901$ $E_{2n+1} = 0$ | $\Gamma_0 = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\Gamma_1 = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ $\Gamma_2 = \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$ $\Gamma_3 = \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$ $\Gamma_4 = \Gamma\left(\frac{9}{2}\right) = \frac{105\sqrt{\pi}}{16}$ $\Gamma_5 = \Gamma\left(\frac{11}{2}\right) = \frac{945\sqrt{\pi}}{32}$ $\Gamma_6 = \Gamma\left(\frac{13}{2}\right) = \frac{10,395\sqrt{\pi}}{64}$ $\Gamma_7 = \Gamma\left(\frac{15}{2}\right) = \frac{135,135\sqrt{\pi}}{128}$ $\Gamma_8 = \Gamma\left(\frac{17}{2}\right) = \frac{2,027,025\sqrt{\pi}}{256}$ $\Gamma_9 = \Gamma\left(\frac{19}{2}\right) = \frac{34,459,425\sqrt{\pi}}{512}$ $\Gamma_{10} = \Gamma\left(\frac{21}{2}\right) = \frac{654,729,075\sqrt{\pi}}{1,024}$ |
| Generating Function | | |
| $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$ | $\frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n$ | $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx$ $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$ $\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{\frac{n-1}{2}}} \sqrt{\pi}$ |
| Recursive Definition | Iterated Sum | Recursive Definition |
| $B_m(n) = n^m - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k(n)}{m-k+1}$ $B_0(n) = 1$ | $E_{2n} = i \sum_{k=1}^{2n+1} \sum_{j=0}^k \binom{k}{j} \frac{(-1)^j (k-2j)^{2n+1}}{2^k i^k k}$ | $\Gamma(n+1) = n \cdot \Gamma(n)$ $\Gamma\left(\frac{n}{2}\right) = \left(\frac{n-2}{2}\right) \cdot \Gamma\left(\frac{n-2}{2}\right)$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ |

Sources

- <https://www.wolframalpha.com>
- <https://en.wikipedia.org>
- <http://ddmf.msr-inria.inria.fr/1.9.1/ddmf>
- <http://web.mit.edu/kenta/www/three/taylor.html>

See Also

- [Harold's Taylor Series Cheat Sheet](#)
- [Harold's Infinite Products Cheat Sheet](#)