Harold's Logic Cheat Sheet

1 December 2025

The 7 Basic Logical Symbols

Operator	Symbol	Example	English
1) Intersection	∧, **∧**, ∧, ⋀, ⋀ •	p ∧ q	• Conjunction • p and q • p, but q • despite the fact that p, q • even though p, q • although p, q • overlap
2) Union	∨, **∨**, ∨, ⋁, ⋁	p ∨ q	• Disjunction • p or q • inclusive or • both combined
3) Negation	¬, ¬, ~	¬p	• not p
4) Conditional	→, →, →, ⟶, ⇒, ⟹, ⊃	p → q	• if p then q • if p, q • q if p • p implies q • p only if q • q in case that p • p is sufficient for q • q is necessary for p
5) Biconditional	↔, ⟷, **↔**, ⇔, ⟺	p ⟷ q	• p iff q • p if and only if q • p is necessary and sufficient for q • if p then q, and conversely • if not p then not q, and conversely
6) Universal Quantifier	∀*x*, (*x*)	∀*x p(x)*	• for all • for any • for each
7) Existential Quantifier	∃*x*	∃*x p(x)*	• there exists • there is at least one
Equivalence (See Biconditional)	≡, ≡, ≡	$expression_1$ ≡ $expression_2$	• is identical to • is equivalent to • is defined as • the two expressions always have the same truth value

"… the structure of all mathematical statements can be understood using these symbols, and all mathematical reasoning can be analyzed in terms of the proper use of these symbols."
Source: "How to Prove It: A Structured Approach", 3rd Edition, p. 75.

Logical Truth Tables

p	q	Conjunction (AND) ∧	NAND ⊼	Disjunction (OR) ∨	NOR ⊽	XOR ⊻, ⊕	XNOR ⊙	Negation (NOT) ¬P
F	**F**	F	T	F	T	F	T	
F	**T**	F	T	T	F	T	F	T
T	**F**	F	T	T	F	T	F	F
T	**T**	T	F	T	F	F	T	

p	q	Material Implication (If … Then) →	Biconditional (Iff) ↔	Tautology (True) ⊤	Contradiction (False) ⊥
F	**F**	T	T	T	F
F	**T**	T	F	T	F
T	**F**	F	F	T	F
T	**T**	T	T	T	F

Blank Truth Tables

Inputs				Output		
p	**q**	**r**	**s**	**x**	**y**	**z**
F	**F**	**F**	**F**			
F	**F**	**F**	**T**			
F	**F**	**T**	**F**			
F	**F**	**T**	**T**			
F	**T**	**F**	**F**			
F	**T**	**F**	**T**			
F	**T**	**T**	**F**			
F	**T**	**T**	**T**			
T	**F**	**F**	**F**			
T	**F**	**F**	**T**			
T	**F**	**T**	**F**			
T	**F**	**T**	**T**			
T	**T**	**F**	**F**			
T	**T**	**F**	**T**			
T	**T**	**T**	**F**			
T	**T**	**T**	**T**			

Inputs			Output	
p	**q**	**r**	**x**	**y**
F	**F**	**F**		
F	**F**	**T**		
F	**T**	**F**		
F	**T**	**T**		
T	**F**	**F**		
T	**F**	**T**		
T	**T**	**F**		
T	**T**	**T**		

Inputs		Output
p	**q**	**x**
F	**F**	
F	**T**	
T	**F**	
T	**T**	

Logical Conditional Connective Laws

Law or Statement	Logical Expression	Is Equivalent To (≡)	Description
Antecedent / Consequent	If <Antecedent> then <Consequent> . <Consequent> if <Antecedent> .		The Antecedent immediately follows the "if" statement.
Conditional Laws	$p \to q$	$\neg p \lor q$ $\neg(p \land \neg q)$ Logical Equivalences: $p \lor q \equiv \neg p \to q$ $p \land q \equiv \neg(p \to \neg q)$ $\neg(p \to q) \equiv p \land \neg q$ $(p \to q) \land (p \to r) \equiv p \to (q \land r)$ $(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$ $(p \to r) \land (q \to r) \equiv (p \land q) \to r$ $(p \to r) \lor (q \to r) \equiv (p \lor q) \to r$	Conditional, If ... Then, Implication
Biconditional Laws	$p \leftrightarrow q$	$(p \to q) \land (q \to p)$ $(p \to q) \land (\neg p \to \neg q)$ $(p \land q) \lor (\neg p \land \neg q)$ $\neg p \leftrightarrow \neg q$ Logical Equivalences: $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	Bi-conditional, If and only If, iff, XNOR
Sufficient Condition	p is a sufficient condition for q	The truth of p suffices to guarantee the truth of q.	
Necessary Condition	q is a necessary condition for p	For p to be true, it is necessary for q to be true also. $\neg q \to \neg p$	
Equivalence	$p \leftrightarrow q$	$p \equiv q$ $p \Longrightarrow q$	Is logically equivalent to $(p \equiv \neg \neg p)$ Is equivalent to
Contrapositive	$p \to q$	$\equiv \neg q \to \neg p$	True
Converse*	$p \to q$	$\not\equiv q \to p$	False
Inverse*	$p \to q$	$\not\equiv \neg p \to \neg q$	False

Rules of Implication
(Inference with Propositions)

Rule Name	Rule Logic	Example
Hypothesis	Givens. First lines of a proof.	It is raining today. You live in McKinney, Texas.
Therefore	$\therefore$	Therefore. In conclusion.
1) Modus Ponens (MP)	p $p \to q$ $\therefore q$	It is raining today. If it is raining today, I will not ride my bike to school. Therefore, I will not ride my bike to school.
2) Modus Tollens (MT)	$\neg q$ $p \to q$ $\therefore \neg p$	If Sam studied for his test, then Sam passed his test. Sam did not pass his test. Therefore, Sam did not study for his test.
3) Hypothetical Syllogism (HS) (Transitivity)	$p \to q$ $q \to r$ $\therefore p \to r$	If you are mad, then you will yell. If you yell, then you will wake the baby. Therefore, if you are mad, then you will wake the baby.
4) Disjunctive Syllogism (DS) (Elimination)	$p \lor q$ $\neg p$ $\therefore q$	Sam studied for his test, or Sam took a nap. Sam did not study for his test. Therefore, Sam took a nap.
5) Constructive Dilemma (CD)	$p \lor q$ $(p \to r) \land (q \to s)$ $\therefore r \lor s$	Oscar is either a dog or a cat. If Oscar is a dog, then you'll have fleas, and if Oscar is a cat, then you'll have fur balls. Therefore, you'll have either fleas or fur balls.
6) Simplification (Simp) (Specialization)	$p \land q$ $\therefore p$	It is rainy today, and it is windy today. Therefore, it is rainy today.
7) Conjunction (Conj)	p q $\therefore p \land q$	Sam studied for his test. Sam passed his test. Therefore, Sam studied for his test, and Sam passed his test.
8) Addition (Add) (Generalization)	p $\therefore p \lor q$	It is raining today. Therefore, it is either It is raining today or snowing today or both.
9) Resolution	$p \lor q$ $\neg p \lor q$ $\therefore q \lor r$	Your shirt is red, or your pants are blue. Your shirt is not red, or your pants are blue. Therefore, your pants are blue, or your shoes are white.

10) Proof by Division into Cases	$p \lor q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore r$	It is raining, or it is Monday. It is raining, so it is wet. It is Monday, so it is wet. It is wet.
11) Contradiction Rule	$\neg p \rightarrow F$ $\therefore p$	If it is not raining is a false statement; then it is raining.

Rules of Replacement

(Logical Connective Laws / Equivalences / Inference)

Law	Union Example	Intersection Example
12) Identity Laws	p ∨ F ≡ p	p ∧ T ≡ p
13) Domination or Null (Universal Bound Laws)	p ∨ T ≡ T	p ∧ F ≡ F
14) Idempotent Laws	p ∨ p ≡ p	p ∧ p ≡ p
15) Double Negations (DN) (Involution Law)	¬ ¬p ≡ p	
16) Negation or Complement (Complementary Laws)	p ∨ ¬p ≡ T ¬F ≡ T	p ∧ ¬p ≡ F ¬T ≡ F
17) Commutative Laws (Com)	p ∨ q ≡ q ∨ p	p ∧ q ≡ q ∧ p
18) Associative Laws (Assoc)	(p ∨ q) ∨ r ≡ p ∨ (q ∨ r)	(p ∧ q) ∧ r ≡ p ∧ (q ∧ r)
19) Distributive Laws (Dist)	p ∧ (q ∨ r) ≡ (p ∧ q) ∨ (p ∧ r)	p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)
20) Uniting Laws	(p ∧ q) ∨ (p ∧ ¬q) ≡ p	(p ∨ q) ∧ (p ∨ ¬q) ≡ p
21) Absorption Laws	p ∨ (p ∧ q) ≡ p	p ∧ (p ∨ q) ≡ p
22) De Morgan's Laws (DM) (Propositional Logic)	p ∨ q ≡ ¬(¬p ∧ ¬q) ¬(p ∨ q) ≡ ¬p ∧ ¬q (p ∨ ¬q) → r ≡ ¬r → (¬p ∧ q)	p ∧ q ≡ ¬(¬p ∨ ¬q) ¬(p ∧ q) ≡ ¬p ∨ ¬q
23) Multiplying and Factoring Laws	(p ∨ q) ∧ (¬p ∨ r) ≡ (p ∧ r) ∨ (¬p ∧ q)	(p ∧ q) ∨ (¬p ∧ r) ≡ (p ∨ r) ∧ (¬p ∨ q)
24) Consensus Laws	(p ∧ q) ∨ (q ∧ r) ∨ (¬p ∧ r) ≡ (p ∧ q) ∨ (¬p ∧ r)	(p ∨ q) ∧ (q ∨ r) ∧ (¬p ∨ r) ≡ (p ∨ q) ∧ (¬p ∨ r)
25) Tautology Laws (⊤)	p ∨ (⊤) ≡ ⊤ p ∨ ¬p ≡ ⊤ (True)	p ∧ (⊤) ≡ p
	¬(⊤) = ⊥	
26) Contradiction Laws (⊥)	p ∨ (⊥) ≡ p	p ∧ (⊥) ≡ ⊥ p ∧ ¬p ≡ ⊥ (False)
	¬(⊥) ≡ ⊤	
27) Exclusive Or Laws (⨁)	p ⨁ q ≡ (p ∨ q) ∨ ¬(p ∧ q)	p ⨁ q ≡ (¬p ∧ q) ∨ (p ∨ ¬q)

Proof Methods

<table>
<tr><th>Method</th><th>Definition</th></tr>
<tr><td>Direct (DP)</td><td>Assume the hypothesis is true, then use logical steps to arrive at the conclusion.
Assume p is true, then conclude q.</td></tr>
<tr><td>Indirect (IP)
(Contradiction)</td><td>Assume the opposite of what you want to prove, then show this leads to a contradiction.
To prove p, assume $\neg p$ and derive a contradiction, such as $\neg q \land q$.
If some statement is assumed true, and a logical contradiction occurs, then the statement must be false.
Can also be a proof by counterexample.
E.g., Assume $\neg(p \rightarrow q)$, which is equivalent to $p \land \neg q$.
Assumption for Indirect Proof (AIP)</td></tr>
<tr><td>Conditional (CP)</td><td>Assume a hypothesis <u>temporarily</u> is true to derive a conclusion.
Assume p, derive q; conclude $p \rightarrow q$.
The goal is not to prove p is true in reality, but to prove that if p were true, then q would necessarily follow.
Assumption for Conditional Proof (ACP)</td></tr>
<tr><td>Contrapositive</td><td>Modus Tollens.
Infers the statement $p \rightarrow q$ by establishing the logically equivalent contrapositive statement: $\neg q \rightarrow \neg p$.
When given $p \rightarrow q$, assume $\neg q$ is true, then prove $\neg p$.
We prove that if the negation of the original conclusion is false, then the negation of the initial theorem is false.
Relies on De Morgen's Law.
<table>
<tr><th>p</th><th>q</th><th>p → q</th><th>Technique</th></tr>
<tr><td>F</td><td>F</td><td>T</td><td>Modus Tollens</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td>(seems forced)</td></tr>
<tr><td>T</td><td>F</td><td>F</td><td></td></tr>
<tr><td>T</td><td>T</td><td>T</td><td>Modus Ponens</td></tr>
</table>
A proof by contrapositive is a special case of a proof by contradiction (indirect).</td></tr>
<tr><td>Construction
(Example)</td><td>The construction of a concrete example with a property to show that something having that property exists.
AKA proof by example.</td></tr>
<tr><td>Exhaustion / By Cases</td><td>The conclusion is established by dividing it into a finite number of cases and proving each one separately.</td></tr>
<tr><td>Induction</td><td>A single "base case" is proved, and an "induction rule" is proved that establishes that any arbitrary case implies the next case.</td></tr>
</table>

Logical Predicates

Definition	Logical Expression	Is Equivalent To (≡)	Plain English
Universe of Discourse	***U***	All possible inputs in a given range	• Universe of Discourse • Universal Set • Universe
Domain of Discourse	$\mathbb{D}$	All possible inputs in a given range	• Domain of Discourse • Universe of Discourse
Proposition or Logical Statement	*p: "Roxy is a mammal."*	*p*	• Must be True or False • Cannot be a question • Cannot be a command
Predicate	*P(x): "x is a mammal"*	*P(x)*	• A logical statement whose truth value is a function of one or more variables • Truth depends upon the input variable *x* • P(x) ≠ a number • P(5) is a proposition
Example Statements	*q:* $\forall x \in \mathbb{D}$, *P(x)*: "x is a mammal"	*"For all x in the domain of discourse, P(x) is a mammal."*	• Is either True or False • A quantified predicate turns it into a logical statement
	T(x, y)	"*x* is a twin of *y*."	Predicate with two input variables
Truth Set (Single Free Variable)	*T = P(x)*	$T = \{a \mid P(a)\}$ $T = \{a \in A \mid P(a)\}$ $a \in T$	The set of all values of x that make the statement *p(x)* true
	Example:	$P(x_1)$, $P(x_2)$, and $P(x_3)$ are True	
Truth Set (Ordered Pair)	*T = P(x, y)*	$\{(a, b) \in A \times B \mid P(a, b)\}$ $(a, b) \in T$	Cross product truth set
	Examples:	$\{(p, n) \in P \times \mathbb{N} \mid$ the person p has n children$\}$ = {(John, 2), …} $\{(p, c, n) \in P \times C \times \mathbb{N} \mid$ the person p has lived in the city c for n years$\}$	

Logical Quantifiers

Definition	Logical Expression	Is Equivalent To (≡)	Plain English
Universal Quantifier (∀)	$\forall x\ P(x)$ $\forall x \in P(x)$ $\forall x \in \mathbb{D},\ P(x)$ *∀x, if x is in* $\mathbb{D}$ *then P(x)*	"For all *x* in the domain, *P(x)* is true" $\forall x \in A\ P(x) \equiv \forall x\ (x \in A \rightarrow P(x))$ For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$, $\forall x\ P(x) \equiv P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_k)$	• for all • all elements • for each member • any • anyone • anything • every • everyone • everybody • everything • x could be anything at all • whoever
Existential Quantifier (∃)	$\exists x\ P(x)$ $\exists x \in P(x)$ $\exists x \in \mathbb{D},\ P(x)$	"There exists x in the domain, such that *P(x)* is true" For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$, $\exists x\ P(x) \equiv P(a_1) \vee P(a_2) \vee \ldots \vee P(a_k)$ $P(x) \neq \emptyset$	• there exists an x • there is • some • someone • somebody • something • at least one value of x • there is at least one x • it is the case that • the truth set is not equal to ∅ • a few
Uniqueness Quantifier (∃!)	$\exists!x\ P(x)$	there is a unique x in *P(x)* such that … $\exists x\ (P(x) \wedge \neg\ y\ (P(y) \wedge y \neq x))$ $\exists x\ (P(x) \wedge \forall y\ (P(y) \rightarrow y = x))$ $\exists x\ \forall y\ (P(y) \leftrightarrow y = x)$ $\exists x\ P(x) \wedge \forall y\ \forall z((P(y) \wedge P(z)) \rightarrow y = z)$	• unique • there is a unique x • there exists exactly one • there is exactly one x such that *P(x)*
Negated Existential Quantifier	$\neg\ [\exists x\ P(x)]$	$\forall x\ \neg P(x)$	• nobody • no one • not one • there does not exist
	$\neg\ [\forall x\ P(x)]$	$\exists x\ \neg P(x)$	
Order of Precedence	PEMDAS for Logic: 1. Parenthesis 2. Logical NOT 3. Quantifiers 4. Logical AND 5. Logical OR 6. Logical Conditional 7. Logical Biconditional	Symbol: (), {}, [] ¬ ∀, ∃ ∧ ∨ → ↔	Applied Left to Right Example : $\forall x\ P(x) \wedge Q(x) \equiv (\forall x\ P(x)) \wedge Q(x)$

Quantifier Logic Examples

Quantifier	Symbolic Translations	English Example
Everyone	$\forall x\ P(x)$	• Everyone is <something>.
	$\forall x\ \forall y\ P(x, y)$ NOTE: includes (x = y)	• Everyone <did something> to everyone.
Everyone Else	$\forall x\ (x \neq y \rightarrow P(x))$	• Everyone except <someone> is <something>.
	$\forall x\ \forall y\ (x \neq y) \rightarrow P(x, y)$ NOTE: excludes (x = y)	• Everyone <did something> to everyone else.
Not Every	$\neg\forall x\ (A(x) \rightarrow B(x))$	• It is not the case that every <something> <did something>.
	$\exists x\ \neg P(x)$	• Not every <something> <did something>.
Someone Else	$\exists x\ ((x \neq y)- \lor P(x))$	• Someone other than <someone> is <something>.
	$\forall x\ \exists y\ ((x \neq y) \land P(x, y))$ NOTE: excludes (x = y)	• Everyone <did something> to <someone> else.
Exactly One	$\exists!x\ P(x)$	• Exactly one person <did something>.
	$\exists x\ (P(x) \land \forall y\ ((x \neq y) \rightarrow \neg P(y))) \equiv$	
No One	$\neg\exists x\ P(x)$	• No one <did something>.
	$\forall x\ \neg P(x)$	

Rules of Inference with Quantifiers

Rule Name	Rule Logic	English Example
Variables	**x** : Quantified variable	The domain is the set of all integers.
Elements	**c, d** : Elements of the domain, arbitrary or particular	c is a particular integer. Element definition.
Universal Instantiation (UI)	*c* is an element (arbitrary or particular) $\forall x\, P(x)$ $\therefore P(c)$	Sam is a student in the class. Every student in the class completed the assignment. Therefore, Sam completed his assignment.
Universal Generalization (UG)	*c* is an arbitrary element $P(c)$ $\therefore \forall x\, P(x)$	All psychiatrists are doctors. All doctors are college graduates. Therefore, all psychiatrists are college graduates. 1. $\forall x\, (P(x) \rightarrow D(x))$ Given 2. $\forall x\, (D(x) \rightarrow C(x))$ Given 3. $P(x) \rightarrow D(x)$ 1, UI 4. $D(x) \rightarrow C(x)$ 2, UI 5. … 6. $P(x) \rightarrow C(x)$ 7. $\therefore \forall x\, (P(x) \rightarrow C(x))$ 6, UG
Existential Instantiation (EI)	$\exists x\, P(x)$ $\therefore$ (*c* is a particular element) • $P(c)$	All attorneys are college graduates. Some attorneys are golfers. Therefore, some golfers are college graduates. i.e., If an object is known to exist, then that object can be given a name.
Existential Generalization (EG)	*c* is an element (arbitrary or particular) $P(c)$ $\therefore \exists x\, P(x)$	All tenors are singers. Andrea Bocelli is a tenor. Therefore, there is at least one singer. 1. $\forall x\, (T(x) \rightarrow S(x))$ Given 2. $T(a)$ Given 3. $T(a) \rightarrow S(a)$ 1, UI 4. $S(a)$ 2,3, MP 5. $\therefore \exists x\, S(x)$ 4, EG

Quantifier Translation Hints

Statement Form	Symbolic Translation	English Example
A B	$A(x) \land B(x)$	Pretty girl.
Either A or B	$A(x) \lor B(x)$	Rachel is a journalist or a newscaster.
	$(A(x) \lor \neg B(x)) \lor (\neg A(x) \lor B(x))$	Rachel is either a journalist or a newscaster, but not both. (XOR)
Neither A nor B	$\neg A(x) \land \neg B(x)$	Neither Wordsworth nor Shelley was Irish.
A is/are B	$A(x) \rightarrow B(x)$	Eli is a student. Sea lions are mammals.
Anything is A	$\forall x\, A(x)$	Anything is conceivable.
All A are B	$\forall x\, (A(x) \rightarrow B(x))$	All maples are trees.
Some A are B	$\exists x\, (A(x) \land B(x))$	Some grapes are sour.
Some A are not B	$\exists x\, (A(x) \land \neg B(x))$	Some grapes are not sour.
A exist	$\exists x\, A(x)$	Tigers exist.
A do not exist	$\neg \exists x\, A(x)$ $\forall x\, \neg A(x)$	Unicorns do not exist.
No A are B	$\neg \forall x\, (A(x) \rightarrow B(x))$ $\exists x\, (A(x) \land \neg B(x))$	No novels are biographies.
Not a single A did B	$\forall x\, (A(x) \rightarrow \neg B(x))$ $\neg \exists x\, (A(x) \land B(x))$	Not a single psychologist attended the convention.
Whoever is A is B	$\forall x\, (A(x) \rightarrow B(x))$	Whoever is a socialite is vain.
Some A B iff C	$\exists x\, (A(x) \land (B(x) \equiv C(x)))$	Some dogs bite if and only if they are teased.
Some A B are C	$\exists x\, [(A(x) \land B(x)) \land C(x)]$	Some French restaurants are exclusive.
A B are C	$\forall x\, [(A(x) \land B(x)) \rightarrow C(x)]$	Ripe apples are delicious.
A and B are C D	$\forall x\, [(A(x) \lor B(x)) \rightarrow (C(x) \land D(x))]$	Violins and cellos are stringed instruments.
Only i is F.	$F(i) \land \forall x\, [F(x) \rightarrow (x = i)]$	Only Sally is running.
The only F that is G is i.	$F(i) \land G(i) \land \forall x\, [(F(x) \land G(x)) \rightarrow (x = i)]$	The only instrument that is brass is the trumpet.
No F except i is G.		No plants except for Venus flytraps are carnivorous.
All F except i are G.	$F(i) \land \neg G(i) \land \forall x\, [(F(x) \land (x \neq i)) \rightarrow G(x)]$	All students except Billy are on time.
i is the F that is most so-and-so.	$F(i) \land \forall x\, [(F(x) \land (x \neq i)) \rightarrow$ i is more so-and-so than x]	Rex is the dog that is most loved than the rest.
There is at most one F.	$\forall x\, \forall y\, [(F(x) \land F(y)) \rightarrow (x = y)]$	There is at most one moon.
There are at least two F's.	$\exists x\, \exists y\, [F(x) \land F(y) \land (x \neq y)]$	There are at least two moons.
There are exactly two F's.	$\exists x\, \exists y\, \{ F(x) \land F(y) \land (x \neq y) \land \forall z\, [F(z) \rightarrow ((z = x) \lor (z = y))] \}$	There are exactly two moons.
The F is G.	$\exists x\, [F(x) \land \forall y\, (F(y) \rightarrow (y = x)) \land G(x)]$	The moon is bright.

Quantifier Laws

Definition	Logical Expression	Is Equivalent To (≡)	Plain English
Abbreviation	$\exists x\,(x \in A \land \neg P(x))$	$\exists x \in A\ \neg P(x)$	Simplification
Expanding Abbreviation	$\forall x \in A\ P(x)$	$\forall x\,(x \in A \rightarrow P(x))$	Complication
1) Quantifier Negation Laws	$\forall x\ \neg P(x)$	$\neg\exists x\ P(x)$	• nobody's perfect
	$\neg\forall x\ P(x)$	$\exists x\ \neg P(x)$	• not everyone is perfect • someone is imperfect
2) Conditional Law	$x \in A \rightarrow P(x)$	$x \notin A \lor P(x)$	$p \rightarrow q \equiv \neg p \lor q$
3) Subset Negation Law	$x \in A$	$\neg(x \notin A)$	Negate then swap ∈ with ∉, or vice versa
4) De Morgan's Law (Quantifier Negation)	$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$ $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$ $\neg\forall x\ \forall y\ P(x, y) \equiv \exists x\ \exists y\ \neg P(x, y)$ $\neg\forall x\ \exists y\ P(x, y) \equiv \exists x\ \forall y\ \neg P(x, y)$ $\neg\exists x\ \forall y\ P(x, y) \equiv \forall x\ \exists y\ \neg P(x, y)$ $\neg\exists x\ \exists y\ P(x, y) \equiv \forall x\ \forall y\ \neg P(x, y)$		De Morgan's Law for single and nested quantifiers
5) Nested / Multiple-Quantified Statements	$\forall x\ \forall y$	$\forall y\ \forall x$	• for all objects x and y, …
	$\exists x\ \exists y$	$\exists y\ \exists x$	• there are objects x and y such that …
	$\forall x\ \exists y\ P(x, y) \not\equiv \exists x\ \forall y\ P(x, y)$		False Counterexample for $x, y \in \mathbb{Z}$: $\forall x\ \exists y\ (x + y = 0) \equiv$ True $\exists x\ \forall y\ (x + y = 0) \equiv$ False
	$\neg(\forall x\ \exists y\ P(x, y))$	$\exists x\ \forall y\ \neg P(x, y)$	Negation of multiple quantified statements
	$\neg(\exists x\ \forall y\ P(x, y))$	$\forall x\ \exists y\ \neg P(x, y)$	
6) Moving Quantifiers	$\forall x\ (P(x) \rightarrow \exists y\ Q(x, y)) \equiv$ $\forall x\ \exists y\ (P(x) \rightarrow Q(x, y))$		You can move a quantifier left if the variable is not used yet

Quantifier Laws

Definition	Logical Expression	Is Equivalent To (≡)		English Example
Abbreviation	$\exists x\,(x \in A \bullet \neg P(x))$	$\equiv$	$\exists x \in A\ \neg P(x)$	Simplification
Expanding Abbreviation	$\forall x \in A\ P(x)$	$\equiv$	$\forall x\,(x \in A \rightarrow P(x))$	Complication
1. Quantifier Negation Laws (QN)	$\forall x\ P(x)$	$\equiv$	$\neg\exists x\ \neg P(x)$	• everyone is perfect • no one is imperfect
	$\exists x\ P(x)$	$\equiv$	$\neg\forall x\ \neg P(x)$	• somebody is perfect • nobody is imperfect
	$\neg\forall x\ P(x)$	$\equiv$	$\exists x\ \neg P(x)$	• not everyone is perfect • someone is imperfect
	$\neg\exists x\ P(x)$	$\equiv$	$\forall x\ \neg P(x)$	• nobody is perfect • everybody is imperfect
2. Conditional Law (ACP)	$x \in A \rightarrow P(x)$	$\equiv$	$x \notin A \vee P(x)$	$p \rightarrow q \equiv \neg p \vee q$
3. Subset Negation Law	$x \in A$	$\equiv$	$\neg(x \notin A)$	Negate then swap ∈ with ∉, or vice versa
4. De Morgan's Law (Quantifier Negation)	$\neg\forall x\ P(x)$	$\equiv$	$\exists x\ \neg P(x)$	De Morgan's Law for a <u>single</u> quantifier
	$\neg\exists x\ P(x)$	$\equiv$	$\forall x\ \neg P(x)$	
	$\neg\forall x\ \forall y\ P(x, y)$	$\equiv$	$\exists x\ \exists y\ \neg P(x, y)$	De Morgan's Law for <u>nested</u> quantifiers
	$\neg\forall x\ \exists x\ P(x, y)$	$\equiv$	$\exists x\ \forall y\ \neg P(x, y)$	
	$\neg\exists x\ \forall y\ P(x, y)$	$\equiv$	$\forall x\ \exists y\ \neg P(x, y)$	
	$\neg\exists x\ \exists y\ P(x, y)$	$\equiv$	$\forall x\ \forall y\ \neg P(x, y)$	
5. Nested / Multiple-Quantified Statements	$\forall x\ \forall y$	$\equiv$	$\forall y\ \forall x$	• for all objects x and y, …
	$\exists x\ \exists y$	$\equiv$	$\exists y\ \exists x$	• there are objects x and y such that …
	$\forall x\ \exists y\ P(x, y)$	$\not\equiv$	$\exists x\ \forall y\ P(x, y)$	False Counterexample for $x, y \in \mathbb{Z}$: $\forall x\ \exists y\ (x + y = 0) \equiv$ True $\exists x\ \forall y\ (x + y = 0) \equiv$ False
	$\neg\,(\forall x\ \exists y\ P(x, y))$	$\equiv$	$\exists x\ \forall y\ \neg P(x, y)$	Negation of multiple-quantified statements
	$\neg(\exists x\ \forall y\ P(x, y))$	$\equiv$	$\forall x\ \exists y\ \neg P(x, y)$	
6. Moving Quantifiers	$\forall x\ (P(x) \rightarrow \exists y\ Q(x, y))$	$\equiv$	$\forall x\ \exists y\ (P(x) \rightarrow Q(x, y))$	You can move a quantifier left if the variable is not used yet

Valid Quantifier Formulas

A		B
∀x (*P(x)* ∧ *Q(x)*)	≡	∀x *P(x)* ∧ ∀x *Q(x)*
∃x (*P(x)* ∧ *Q(x)*)	→	∃x *P(x)* ∧ ∃x *Q(x)*
∀x (*P(x)* ∨ *Q(x)*)	←	∀x *P(x)* ∨ ∀x *Q(x)*
∃x (*P(x)* ∨ *Q(x)*)	≡	∃x *P(x)* ∨ ∃x *Q(x)*
∀x (*P(x)* → *Q(x)*)	←	∃x *P(x)* → ∀x *Q(x)*
∃x (*P(x)* → *Q(x)*)	≡	∀x *P(x)* → ∃x *Q(x)*
∀x ¬*P(x)*	≡	¬∃x *P(x)*
∃x ¬*P(x)*	≡	¬∀x *P(x)*
∀x ∃y *T(x, y)*	←	∃y ∀x *T(x, y)*
∀y ∃x *T(x, y)*	←	∃x ∀y *T(x, y)*
∀x ∀y *T(x, y)*	≡	∀y ∀x *T(x, y)*
∃x ∃y *T(x, y)*	≡	∃y ∃x *T(x, y)*
∀x (*P(x)* ∨ *R*)	≡	∀x *P(x)* ∨ *R*
∃x (*P(x)* ∧ *R*)	≡	∃x *P(x)* ∧ *R*
∀x (*P(x)* → *R*)	≡	∃x *P(x)* → *R*
∃x (*P(x)* → *R*)	→	∀x *P(x)* → *R*
∀x (*R* → *Q(x)*)	≡	R → ∀x *Q(x)*
∃x (*R* → *Q(x)*)	→	R → ∃x *Q(x)*
∀x *R*	←	*R*
∃x *R*	→	*R*

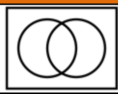
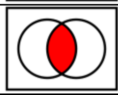
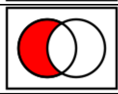
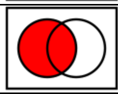
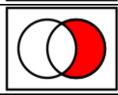
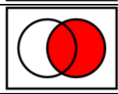
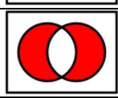
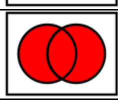
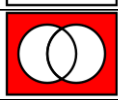
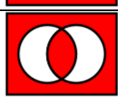
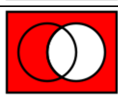
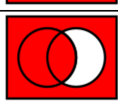
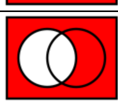
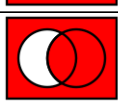
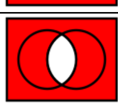
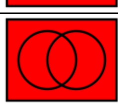
Note: The above formulas are valid in classical first-order logic assuming that *x* does not occur free in *R*.

Invalid Quantifier Formulas

A		B	Counterexample
∃x (*P(x)* ∧ *Q(x)*)	←	∃x *P(x)* ∧ ∃x *Q(x)*	D = {*a*, *b*}, M = {*P(a)*, *Q(b)*}
∀x (*P(x)* ∨ *Q(x)*)	→	∀x *P(x)* ∨ ∀x *Q(x)*	D = {*a*, *b*}, M = {*P(a)*, *Q(b)*}
∀x (*P(x)* → *Q(x)*)	→	∃x *P(x)* → ∀x *Q(x)*	D = {*a*, *b*}, M = {*P(a)*, *Q(a)*}
∀x ∃y *T(x, y)*	→	∃y ∀x *T(x, y)*	D = {*a*, *b*}, M = {*T(a, b)*, *T(b, a)*}
∃x (*P(x)* → *R*)	←	∀x *P(x)* → *R*	D = Ø, M = {*R*}
∃x (*R* → *Q(x)*)	←	*R* → ∃x *Q(x)*	D = Ø, M = Ø
∀x *R*	→	*R*	D = Ø, M = Ø
∃x *R*	←	*R*	D = Ø, M = {*R*}

Note: if empty domains are not allowed, then the last four implications above are in fact valid.

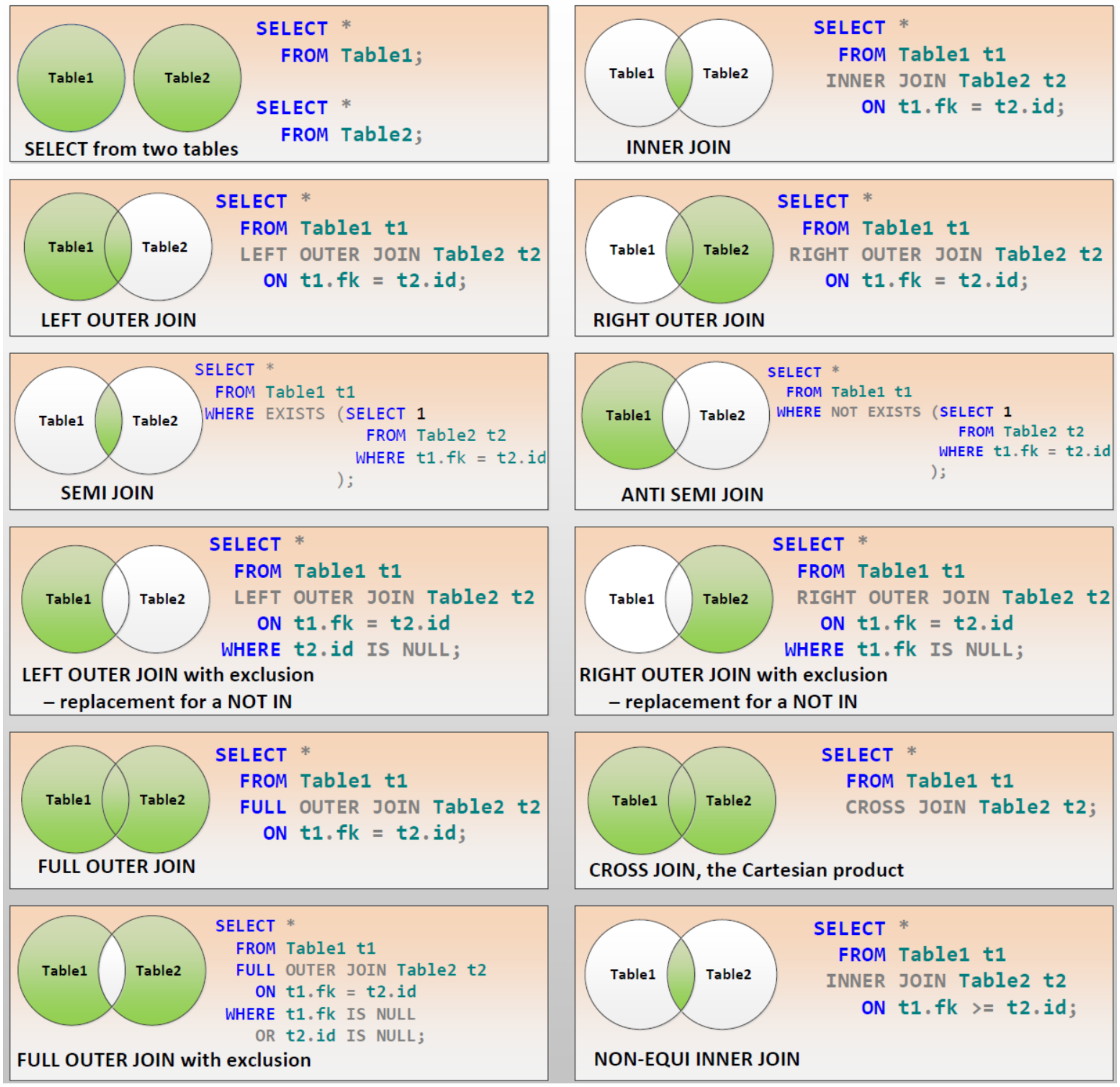
The Sixteen Logical Operations on Two Variables

#	Venn	Sym	Logical Notation(s)	Name(s)
0000		⊥	0	Contradiction; falsehood; antilogy; constant 0
0001		∧	$x \wedge y, \quad xy, \quad x \,\&\, y$	Conjunction; AND
0010		$\bar{\supset}$	$x \wedge \bar{y}, x \not\supset y, [x > y], x \mathbin{\dot{-}} y$	Nonimplication; difference; but not
0011		L	x	Left projection
0100		$\bar{\subset}$	$\bar{x} \wedge y, x \not\subset y, [x < y], y \mathbin{\dot{-}} x$	Converse nonimplication; not ... but
0101		R	y	Right projection
0110		⊕	$x \oplus y, \quad x \not\equiv y, \quad x\text{^}y$	Exclusive disjunction; nonequivalence; XOR
0111		∨	$x \vee y, \quad x \mid y$	(Inclusive) disjunction; and/or; OR
1000		⊽	$\bar{x} \wedge \bar{y}, \overline{x \vee y}, x \barvee y, x \downarrow y$	Nondisjunction; joint denial; neither... NOR
1001		≡	$x \equiv y, \quad x \leftrightarrow y, \quad x \Leftrightarrow y$	Equivalence; if and only if; IFF
1010		$\bar{\text{R}}$	$\bar{y}, \quad \neg y, \quad !\,y, \quad \sim y$	Right complementation; NOT
1011		⊂	$x \vee \bar{y}, x \subset y, x \Leftarrow y, [x \geq y], x^y$	Converse implication; IF
1100		$\bar{\text{L}}$	$\bar{x}, \quad \neg x, \quad !\,x, \quad \sim x$	Left complementation; NOT
1101		⊃	$\bar{x} \vee y, x \supset y, x \Rightarrow$ $y, [x \leq y], y^x$	Implication; only if; if … then
1110		⊼	$\bar{x} \vee \bar{y}, \overline{x \wedge y}, x \barwedge y, x \mid y$	Nonconjunction; not both … and; NAND
1111		⊤	1	Affirmation; validity; tautology; constant 1

Donald E. Knuth (1968). 7.1.1 Boolean Basics, *The Art of Computer Programming*, Pre-fascicle 0B: The sixteen logical operations in two variables. See also Wikipedia, Truth function.

The Twelve SQL Join Types

Created by Steve Steadman



Sources

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- https://en.wikipedia.org/wiki/List_of_logic_symbols
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- Knuth, Donald E. (1968). 7.1.1 Boolean Basics, *The Art of Computer Programming*, Pre-fascicle 0B: The sixteen logical operations in two variables.
- Steadman, Steven (2025). TSQL Join Types, version 22.03. https://stedmansolutions.kit.com/1296307228

See Also

- Harold's Logic Cheat Sheet
- Harold's Logic (Philosophy) Cheat Sheet
- Harold's Sets Cheat Sheet
- Harold's Boolean Algebra Cheat Sheet
- Harold's Proofs Cheat Sheet