Harold's Trig Proofs
Cheat Sheet

4 February 2026

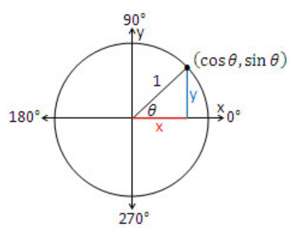
After proving only three (3) trig formulas, we can easily derive **ALL** the trig formulas!

1. Pythagorean Identity
2. Sum and difference formula for sine
3. Sum and difference formula for cosine

A list of Paul Dawkins' trig formulas is provided on the last two pages of this cheat sheet.

Proof of Pythagorean Identities

Proof

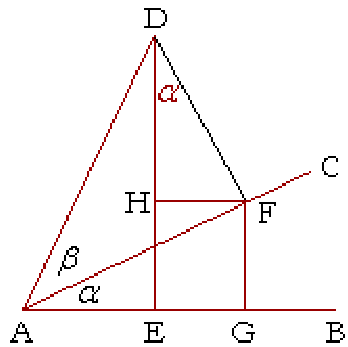
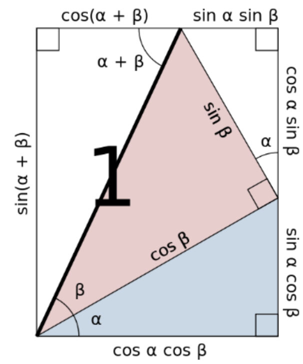
Given 	Pythagorean Theorem $x^2 + y^2 = r^2$ $r = 1$ $\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} = \frac{y}{1} = y$ $\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} = \frac{x}{1} = x$
Substitute and Simplify	$\sin^2\theta + \cos^2\theta = 1^2$
Formula	$\mathbf{\sin^2\theta + \cos^2\theta = 1 \quad [1]}$

Proof

Given	$\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Divide by $\cos^2\theta$, then Simplify	$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$
Formula	$\mathbf{\tan^2\theta + 1 = \sec^2\theta \quad [2]}$

Proof

Given	$\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Divide by $\sin^2\theta$, then Simplify	$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$
Formula	$\mathbf{1 + \cot^2\theta = \csc^2\theta \quad [3]}$

Proof of Sum and Difference Formulas		
Trig Sum and Difference Formulas	$\boldsymbol{\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta}$ $\boldsymbol{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$	
Proof Diagram		

Proof of $sin(\alpha \pm \beta)$	
Prove Sum	
Given	$\sin(\alpha+\beta) = \dfrac{ED}{DA} = \dfrac{opposite}{hypotenuse}$
Alternate interior angles are congruent	$\alpha = \angle CAB = \angle HFA = \angle HDF$
Tallest vertical line	$ED = GF + HD$
Substitute, then divide and multiply by AF & FD	$\sin(\alpha+\beta) = \dfrac{ED}{AD} = \dfrac{GF}{AD} + \dfrac{HD}{AD} = \dfrac{GF}{AF}\dfrac{AF}{AD} + \dfrac{HD}{FD}\dfrac{FD}{AD}$
Convert back to trig formulas	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad [4]$
Prove Difference	
Replace $+\beta$ with $-\beta$	$\cos(-\beta) = \cos(\beta)$ $\sin(-\beta) = -\sin(\beta)$
Simplify	$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad [5]$
General Formula [4+5]	$\boldsymbol{\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad [6]}$

Proof of $cos(\alpha \pm \beta)$	
Prove Sum	
Given	$\cos(\alpha + \beta) = \frac{AE}{AD} = \frac{adjacent}{hypotenuse}$
Longest horizontal line	$EA = GA - FH$
Substitute, then divide and multiply by AF & DF	$\cos(\alpha + \beta) = \frac{EA}{AD} = \frac{GA}{AD} - \frac{FH}{AD} = \frac{GA}{AF}\frac{AF}{AD} + \frac{FH}{DF}\frac{DF}{AD}$
Convert back to trig formulas	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad [7]$
Prove Difference	
Replace $+\beta$ with $-\beta$	$\cos(-\beta) = \cos(\beta)$ $\sin(-\beta) = -\sin(\beta)$
Simplify	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad [8]$
General Formula [7+8]	$\boldsymbol{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \quad [9]}$

Proof of $tan(\alpha \pm \beta)$	
Prove Sum and Difference	
Given	$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$
Substitute	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad [6]$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \quad [9]$
Divide by $(\cos\alpha\cos\beta)$, then Simplify	$\tan(\alpha \pm \beta) = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$
General Formula	$\boldsymbol{\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\ \tan\beta} \quad [10]}$

Proof of Double Angle Formulas (2θ)

Proof

Given	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad [4]$
Substitute	$\theta = \alpha = \beta$
Simplify	$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$
Formula	$\mathbf{\sin(2\theta) = 2\sin\theta\cos\theta \quad [14]}$

Proof

Given	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad [7]$
Substitute	$\theta = \alpha = \beta$
Simplify	$\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$
Formula	$\mathbf{\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]}$

Proof

Given	$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]$ $\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Substitute	$\sin^2\theta = 1 - \cos^2\theta$
Simplify	$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta)$
Formula	$\mathbf{\cos(2\theta) = 2\cos^2\theta - 1 \quad [16]}$

Proof

Given	$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]$ $\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Substitute	$\cos^2\theta = 1 - \sin^2\theta$
Simplify	$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta$
Formula	$\mathbf{\cos(2\theta) = 1 - 2\sin^2\theta \quad [17]}$

Proof

Given	$\tan(2\theta) = \dfrac{\sin(2\theta)}{\cos(2\theta)}$
Substitute	$\sin(2\theta) = 2\sin\theta\cos\theta \quad [14]$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]$
Divide by $\cos^2\theta$	$\tan(2\theta) = \dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$
Simplify	$\tan(2\theta) = \dfrac{\left(\frac{2\sin\theta\cos\theta}{\cos^2\theta}\right)}{\left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}\right)}$
Formula	$\mathbf{\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta} \quad [18]}$

Proof of Half Angle Formulas (θ/2)

Proof

Given	$\cos(2\theta) = 1 - 2\sin^2\theta \quad [17]$
Solve for $\sin^2\theta$	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2} \quad [19a]$
Substitute	$\theta = \dfrac{\theta}{2}$
Solve	$\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos(\theta)}{2}$
Formula	$\mathbf{\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\theta)}{2}} \quad [19b]}$

Proof

Given	$\cos(2\theta) = 2\cos^2\theta - 1 \quad [16]$
Solve for $\cos^2\theta$	$\cos^2\theta = \dfrac{1+\cos(2\theta)}{2} \quad [20a]$
Substitute	$\theta = \dfrac{\theta}{2}$
Solve	$\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1+\cos(\theta)}{2}$
Formula	$\mathbf{\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos(\theta)}{2}} \quad [20b]}$

Proof

Given	$\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$
Substitute	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2} \quad [19a]$ $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2} \quad [20a]$
Simplify	$\tan^2\theta = \dfrac{\left(\dfrac{1-\cos(2\theta)}{2}\right)}{\left(\dfrac{1+\cos(2\theta)}{2}\right)} = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Substitute	$\theta = \dfrac{\theta}{2}$
Solve	$\tan^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos(\theta)}{1+\cos(\theta)} \quad [21a]$
Formula	$\mathbf{\tan\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\theta)}{1+\cos(\theta)}} \quad [21b]}$

Proof of Cofunction Formulas

Proof

Given	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \quad [5]$
Substitute	$\alpha=\frac{\pi}{2}, \beta=\theta$
Simplify	$\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)\cos\theta+\cos\left(\frac{\pi}{2}\right)\sin\theta$
Formula	$\boldsymbol{\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad [22]}$

Proof

Given	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \quad [8]$
Substitute	$\alpha=\frac{\pi}{2}, \beta=\theta$
Simplify	$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}\right)\cos\theta-\sin\left(\frac{\pi}{2}\right)\sin\theta$
Formula	$\boldsymbol{\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad [23]}$

Proof

Given	$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$
Substitute	$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad [22]$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad [23]$
Simplify	$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{\cos\theta}{\sin\theta}=\cot\theta$
Formula	$\boldsymbol{\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta \quad [24]}$

Proof

Given	$\sec(\theta)=\frac{1}{\cos(\theta)}$
Substitute	$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad [23]$
Simplify	$\sec\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{1}{\sin(\theta)}=\csc\theta$
Formula	$\boldsymbol{\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \quad [25]}$

Proof

Given	$\csc(\theta) = \dfrac{1}{\sin(\theta)}$
Substitute	$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta \quad [22]$
Simplify	$\csc\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \dfrac{1}{\cos(\theta)} = \sec\theta$
Formula	$\mathbf{csc}\left(\dfrac{\boldsymbol{\pi}}{\mathbf{2}} - \boldsymbol{\theta}\right) = \mathbf{sec}\,\boldsymbol{\theta} \quad [\mathbf{26}]$

Proof

Given	$\cot(\theta) = \dfrac{1}{\tan(\theta)}$
Substitute	$\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta \quad [24]$
Simplify	$\cot\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\left(\frac{\pi}{2} - \theta\right)} = \dfrac{1}{\cot(\theta)} = \tan\theta$
Formula	$\mathbf{cot}\left(\dfrac{\boldsymbol{\pi}}{\mathbf{2}} - \boldsymbol{\theta}\right) = \mathbf{tan}\,\boldsymbol{\theta} \quad [\mathbf{27}]$

Additional Resources

- AoPSOnline (2025). Art of Problem Solving, Proofs of trig identities. https://artofproblemsolving.com/wiki/index.php/Proofs_of_trig_identities
- Dawkins, Paul (2005). Trig Cheat Sheet, pp. 1-2. https://www.toomey.org/tutor/pauls_online_math_notes/cheat_sheets_full/Trig_Cheat_Sheet.pdf

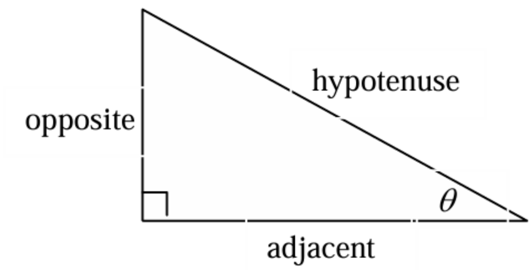
Paul Dawkins Trig Cheat Sheet

Definition of the Trig Functions

Right triangle definition

For this definition we assume that

$0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$.



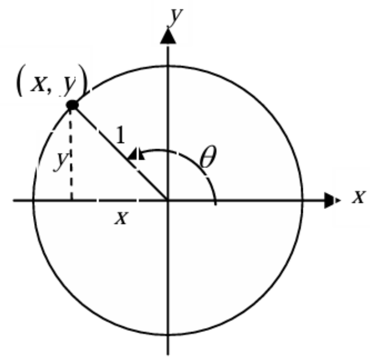
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Unit circle definition

For this definition θ is any angle.



$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin\theta$, θ can be any angle

$\cos\theta$, θ can be any angle

$\tan\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\csc\theta$, $\theta \neq n\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\sec\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\cot\theta$, $\theta \neq n\pi, \quad n = 0, \pm1, \pm2, \ldots$

Range

The range is all possible values to get out of the function.

$-1 \le \sin\theta \le 1 \qquad \csc\theta \ge 1$ and $\csc\theta \le -1$

$-1 \le \cos\theta \le 1 \qquad \sec\theta \ge 1$ and $\sec\theta \le -1$

$-\infty \le \tan\theta \le \infty \qquad -\infty \le \cot\theta \le \infty$

Period

The period of a function is the number, T, such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

$$\csc(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cot(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Periodic Formulas

If n is an integer.

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Half Angle Formulas

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$